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Representations of $S_n \times U(N)$ in repeated tensor products of the unitary groups

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Abstract. The n-fold tensor product space generated by a given irreducible representation of the unitary group $U(N)$ is a representation space for the symmetric group S_n as well as for $U(N)$. Using ideas from the theory of dual pairs, such tensor product spaces are decomposed into irreducible representations of $U(N)$ times reducible representations of S_n. Computationally effective formulae for the multiplicity of irreducible representations of S_n are given. Generating sets of invariant polynomials from the enveloping algebra of $U(N)$ that commute with the S_n action are exhibited; it is shown that the eigenvalues of such operators can be used to break the multiplicity occurring in the dual $U(N) \times S_n$ action.

1. Introduction

A problem of recurring interest in mathematical physics is that of finding the types of irreducible representations of the symmetric group that occur in the decomposition of the n-fold tensor product representations of a given group. In particular, such problems arise when identical representations are coupled together and one wishes to find the symmetric or antisymmetric representations of the symmetric group, corresponding to fermionic or bosonic type systems.

The classic example of such types of problems is the Schur–Weyl duality theorem (cf Weyl 1946 or Želobenko 1973), which states that if the natural representation of the unitary group $U(N)$ (or equivalently, of the general linear group $GL(N, \mathbb{C})$) is tensored n times, then under the joint action of the symmetric group S_n and $GL(N, \mathbb{C})$ this tensor product is decomposed into a direct sum of multiplicity-free irreducible subrepresentations uniquely labelled by Young diagrams. We will give a very simple proof of this result in section 5. What we wish to do in this paper is to give a procedure for explicitly decomposing n-fold tensor product $S_n \times GL(N, \mathbb{C})$ modules of repeated irreducible representations of $GL(N, \mathbb{C})$ with signatures of the form $(M, 0, \ldots, 0)$ and not just the natural representation (signature $(1, 0, \ldots, 0)$). Obviously the decomposition of $S_n \times GL(n, \mathbb{C})$-modules is not multiplicity-free, as the simple example $(4, 0, 0) \otimes (4, 0, 0) \otimes (4, 0, 0)$ in section 5 already shows. To break these multiplicities we shall make use of S_n-invariant differential operators; this conforms with our general method of using invariant differential operators of various groups to resolve the multiplicity problem as shown in Klink and Ton-That (1988, 1989a, b). Section 4 deals with these S_n-invariant differential operators. To carry out our procedure we also make use of the theory of dual pairs as discussed in Moshinsky and Quesne (1970), Howe (1985, 1987), Klink and Ton-That (1988, 1989a, b). The theory of dual pairs provides a natural

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setting for generalising the Schur-Weyl duality type problems, not only for $\mathrm{U}(N)$, but for all the compact groups. The notion of dual pairs is discussed in section 2, and in section 3 we will show how the problem is connected with the notion of representations of the Weyl group of $\mathrm{GL}(n, \mathbb{C})$ on the zero-weight spaces of $\mathrm{GL}(n, \mathbb{C})$-modules. Combining the theory of dual pairs with the notion of zero-weight spaces gives the multiplicity of the representations of the symmetric group S_n (considered as the Weyl group of $\mathrm{GL}(n, \mathbb{C})$) in n-fold tensor products of representations of $\mathrm{GL}(N, \mathbb{C})$ with signature $(M, 0, \ldots, 0)$, but unfortunately all known formulae concerning this multiplicity (cf Gutkin 1973, Kostant 1975, Ariki *et al* 1985) are sufficiently complicated when n becomes so large as to not be very useful computationally. So in section 5 we present a computationally useful way, especially adaptable to computers, of finding symmetric group multiplicities, by showing how to calculate the (generally) reducible characters of the symmetric group S_n and more importantly, how to break these multiplicities using S_n-invariant differential operators. Several examples illustrate the use of our method for calculating characters of S_n. We conclude with a brief discussion on the generalisation of our result to n-fold tensor products of arbitrary irreducible representations of $\mathrm{GL}(N, \mathbb{C})$ of the same signature.

2. Representations of the symmetric group S_n and dual pairs

Let $\mathbb{C}^{n\times N}$ denote the vector space of $n \times N$ complex matrices. Let $\mathrm{d}Z = \prod \mathrm{d}X_{ij}\, \mathrm{d}Y_{ij}$ $(Z_{ij} = X_{ij} + \sqrt{-1}\, Y_{ij},\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant N)$ denote the Lebesgue product measure on $\mathbb{R}^{nN}$. Define a Gaussian measure $\mathrm{d}\mu$ on $\mathbb{C}^{n\times N}$ by

$$\mathrm{d}\mu(Z) = \pi^{-nN} \exp[-\mathrm{tr}(Z\bar{Z}^{\mathrm{t}})]\, \mathrm{d}Z.$$

Let $\mathscr{F} \equiv \mathscr{F}(\mathbb{C}^{n\times N})$ be the Hilbert space of all holomorphic entire functions F on $\mathbb{C}^{n\times N}$ which are square integrable, i.e. $\int_{\mathbb{C}^{n\times N}} |F(Z)|^2\, \mathrm{d}\mu(Z) < \infty$. In Klink and Ton-That (1989b) it is shown that the integration inner product of $\mathscr{F}$ is identical to the computationally more useful differentiation inner product

$$\langle F_1, F_2 \rangle = F_1(D)\overline{F_2(\bar{Z})}|_{Z=0} \tag{2.1}$$

where $F(D)$ denotes the differential operator formally obtained from the infinite series $F(Z)$ by replacing Z_{ij} by $\partial/\partial Z_{ij}$. Such an inner product is useful because, as will be shown, the irreducible representation spaces of $\mathrm{GL}(N, \mathbb{C})$ are given by polynomials in $\mathscr{F}$.

The action of $\mathrm{GL}(N, \mathbb{C})$ on $\mathscr{F}$ is given by right translation

$$[R(g)F](Z) = F(Zg) \qquad g \in \mathrm{GL}(N, \mathbb{C}) \tag{2.2}$$

and is highly reducible. Similarly, $\mathrm{GL}(n, \mathbb{C})$ acts on $\mathscr{F}$ by left translation

$$[L(h')F](Z) = F((h')^{-1}Z) \qquad h' \in \mathrm{GL}(n, \mathbb{C}). \tag{2.3}$$

Let r denote the minimum of n and N and let $(m) = (m_1, \ldots, m_r)$ be an r-tuple of non-negative integers satisfying the dominant condition $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_r$. Then models of irreducible representations of $\mathrm{GL}(N, \mathbb{C})$ (respectively, $\mathrm{GL}(n, \mathbb{C})$) of signature (m) can be realised in $\mathscr{F}$ as follows. Let B_N (respectively, B'_n) denote the Borel subgroup of $\mathrm{GL}(N, \mathbb{C})$ (respectively, $\mathrm{GL}(n, \mathbb{C})$) consisting of upper triangular (respectively, lower triangular) matrices.

Set

$$V^{(m)} = \{F \in \mathscr{F}\colon F(Zb) = b_{11}^{m_1} \ldots b_{rr}^{m_r} F(Z), \forall b \in B_N\}$$
$$V'^{(m)} = \{F \in \mathscr{F}\colon F(Zb) = b_{11}^{m_1} \ldots b_{rr}^{m_r} F(Z), \forall B \in B_N\} \tag{2.4}$$

then the representation of GL(N, $\mathbb{C}$) (respectively, GL(n, $\mathbb{C}$)) obtained by right translation (respectively, left translation) on $V^{(m)}$ (respectively, on $V'^{(m)}$) is irreducible with signature (m).

Now under the joint action $L(h')\otimes R(g)$ of GL(n, $\mathbb{C}$) and GL(N, $\mathbb{C}$) the Fock space $\mathscr{F}$ is decomposed into irreducible submodules

$$\mathscr{F} = \sum_{(m)} \oplus \mathscr{T}^{(m)} \tag{2.5}$$

where (m) ranges over all r-tuples of integers such that $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_r \geqslant 0$ and where $\mathscr{T}^{(m)}$ denotes the isotypic component of (m) in $\mathscr{F}$. The isotypic component $\mathscr{T}^{(m)}$ is the sum of all GL(N, $\mathbb{C}$)-submodules in $\mathscr{F}$ which are isomorphic to $V^{(m)}$; or equivalently, the sum of all GL(n, $\mathbb{C}$)-submodules in $\mathscr{F}$ that are isomorphic to $V'^{(m)}$. It can be shown that $\mathscr{T}^{(m)}$ is generated by $L(h')F$, $h' \in \mathrm{GL}(n, \mathbb{C})$, $F \in V^{(m)}$; or equivalently, by $R(g)F'$, $g \in \mathrm{GL}(N, \mathbb{C})$, $F' \in V'^{(m)}$ (cf Želobenko 1970, chapter VIII or Howe 1987).

Of central interest in this paper is the n-fold tensor product

$$V_N^{(M,0,\ldots,0)} \oplus \ldots \oplus V_N^{(M,0,\ldots,0)}.$$

Since the representation $V_N^{(M,0,\ldots,0)}$ is repeated n times, the symmetric group S_n of permutations on n elements also acts on this tensor product. The natural space for studying the decomposition of this joint action of $\mathrm{S}_n \times \mathrm{GL}(N, \mathbb{C})$ is the Fock space $\mathscr{F}$. Indeed, let D'_n denote the diagonal subgroup of GL(n, $\mathbb{C}$) and define the subspace $\mathscr{P}^{(\underbrace{M,\ldots,M}_{n})}$ of $\mathscr{F}$ by

$$\mathscr{P}^{(M,\ldots,M)} = \{F \in \mathscr{F}\colon F(d'Z) = d_1'^M \ldots d_n'^M F(Z) = |d'|^M F(Z), \forall d' \in D'_n\} \tag{2.6}$$

where $|d'|$ denotes the determinant of d'. It was shown in Klink and Ton-That (1989b) that $\mathscr{P}^{(M,\ldots,M)}$ is isomorphic to $V^{(M,0,\ldots,0)} \otimes \ldots \otimes V^{(M,0,\ldots,0)}$ (n times) with the joint action of $\mathrm{S}_n \times \mathrm{GL}(n, \mathbb{C})$ on $\mathscr{P}^{(M,\ldots,M)}$ given by

$$[L(\sigma)\otimes R(g)]F(Z) = F(\bar{\sigma}^1 Z g) \qquad \sigma \in S_n, g \in \mathrm{GL}(N, \mathbb{C}). \tag{2.7}$$

Our problem can be set in the context of reductive dual pairs as follows. By definition a reductive dual pair of groups (G′, G) acting on $\mathscr{F}$ is a pair of reductive groups whose representations on $\mathscr{F}$ are such that one group is the centraliser of the other and vice-versa (see Howe 1985). Let $\mathrm{G} = \mathrm{GL}(N, \mathbb{C}) \times \ldots \times \mathrm{GL}(N, \mathbb{C})$ (n times); then the action of G on $\mathscr{F}$ is again given by

$$[R(g)F](Z) = F\left(\begin{bmatrix} Z_1 g_1 \\ Z_n g_n \end{bmatrix}\right) \qquad F \in \mathscr{F}$$
$$g = (g_1, \ldots, g_n) \qquad g_i \in \mathrm{GL}(N, \mathbb{C}) \qquad Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix} \tag{2.8}$$

where Z_i denotes the ith row of $Z \in \mathbb{C}^{n\times N}$. Let G′ be the diagonal subgroup D'_n of GL(n, $\mathbb{C}$) then $\mathrm{G}' \approx \mathrm{GL}(1, \mathbb{C}) \times \ldots \times \mathrm{GL}(1, \mathbb{C})$ (n times) and it was shown in Klink and Ton-That (1989b) that G′ and G form a reductive dual pair on $\mathscr{F}$ such that under the joint action of the pair (G', G) $\mathscr{F}$ is decomposed into isotypic components $\mathscr{F} = \sum_{(M_1,\ldots,M_n)} \oplus \mathscr{P}^{(M_1,\ldots,M_n)}$. If we set $H = \{(g, \ldots, g) \in G\} \approx GL(N, \mathbb{C})$ and $H' = GL(n, \mathbb{C})$ then (H', H) forms another dual pair and we have the decomposition of $\mathscr{F}$ into isotypic components $\mathscr{F}^{(m)}$ from (2.5).

Now the semidirect product $K' = S_n \circledS G'$ is a subgroup of H' such that $G' \subset K' \subset H'$; if there were a dual group K to K' such that $G \subset K \subset H$ then the decomposition of the joint action of the dual pair (K', K) on $\mathscr{P}^{(M,\ldots,M)}$ would give an exact analogue to the Schur–Weyl duality theorem, in the sense that the isotypic components of (K', K) would tie together each irreducible representation label of $GL(N, \mathbb{C}) \subset K$ with an irreducible label of $S_n \subset K'$. Unfortunately, it is not difficult to show that there is no group K between G and H which is dual to K'. This shows that a generalisation of the Schur–Weyl duality theorem is not possible. However, we still have the interesting problem of explicitly decomposing the generalised Schur–Weyl $S_n \times GL(N, \mathbb{C})$-module $\mathscr{P}^{(M,\ldots,M)}$. For this we shall make use of the dual pair action (H', H) to compute the multiplicity of each irreducible $S_n \times GL(N, \mathbb{C})$-module in $\mathscr{P}^{(M,\ldots,M)}$ and the S_n-invariant differential operators on $\mathscr{P}^{(M,\ldots,M)}$ to break this multiplicity.

Now, from Klink and Ton-That (1989b) we can find a complete decomposition of $\mathscr{P}^{(M,\ldots,M)}$ under the single action of $GL(N, \mathbb{C})$, and hence we can write $\mathscr{P}^{(M,\ldots,M)} = \sum_{(M)} \oplus \mathscr{T}^{(m)} \cap \mathscr{P}^{(M,\ldots,M)}$ where the $\mathscr{T}^{(m)}$ are given in (2.5). Obviously the actions of $S_n \subset H' = GL(n, \mathbb{C})$ and $H = GL(N, \mathbb{C})$ commute and leave each $\mathscr{T}^{(m)} \cap \mathscr{P}^{(M,\ldots,M)}$ invariant. Hence, to decompose the $S_n \times GL(N, \mathbb{C})$-module $\mathscr{P}^{(M,\ldots,M)}$ it suffices to decompose each submodule $\mathscr{T}^{(m)} \cap \mathscr{P}^{(M,\ldots,M)}$. But, as we observed before, $\mathscr{T}^{(m)}$ is generated by $L(h')F$, $h' \in GL(n, \mathbb{C})$, $F \in V'^{(m)}$; therefore, we need to find how to send an element of $V^{(m)}$ into $\mathscr{P}^{(M,\ldots,M)}$. This is most easily accomplished by introducing differential operators acting on $\mathscr{F}$ which form a faithful representation of the Lie algebra of $GL(n, \mathbb{C})$:

$$L_{\alpha\beta} = \sum_{i=1}^{N} Z_{\alpha i} \frac{\partial}{\partial Z_{\beta i}} \qquad 1 \leq \alpha, \beta \leq n. \tag{2.9}$$

The elements $L_{\alpha\alpha}$ form a basis for the Lie algebra of the diagonal subgroup $D_n = G'$ of $H' = GL(n, \mathbb{C})$. Further, the condition

$$F(b'Z) = F(Z) \qquad b' = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

needed for F to be an element of $V'^{(m)}$ can be written infinitesimally as

$$L_{\alpha\beta} F = 0 \qquad \forall \alpha < \beta. \tag{2.10}$$

The operators $L_{\alpha\beta}$, $\alpha > \beta$ act as 'lowering operators' and take elements of $V'^{(m)}$ into $\mathscr{P}^{(M,\ldots,M)}$.

If $F \in V^{(m)}$ is chosen to be a highest-weight vector $F^{(m)}_{\mathrm{Max}}$ of $V^{(m)}$ then $F^{(m)}_{\mathrm{Max}}$ is also a highest vector of $V'^{(m)}$ and we can write $F^{(m)}_{\mathrm{Max}}$ as

$$F^{(m)}_{\mathrm{Max}}(Z) = (\Delta^1_1(Z))^{m_1 - m_2} (\Delta^{12}_{12}(Z))^{m_2 - m_3} \ldots (\Delta^{12\ldots r}_{12\ldots r}(Z))^{m_r} \tag{2.11}$$

where $\Delta^1_i(Z) = Z_{1i}$ and $(\Delta^{1\ldots i}_{1\ldots i})(Z)$ denotes the ith principal minor of Z; $1 \leq i \leq r$. Moreover, under the action of the lowering operators, the images of $F^{(m)}_{\mathrm{Max}}$ span the weight space of $V'^{(m)}$ of weight $(M, \ldots, M)$. In fact, if $[k]$ is a Gelfand tableau of the form

$$[k] = \begin{bmatrix} m_{1n} & \dots & m_{rn} 0 & \dots \\ & m_{1,n-1}, & \dots & \\ & \ddots & \cdot^{\cdot^{\cdot}} & \\ & & m_{11} & \end{bmatrix}$$

where $m_{in} = m_i$, $1 \leq i \leq r$, then it was shown in Klink and Ton-That (1989b) that $[k]$ labels a weight vector with weight $(M, \dots, M)$ if and only if

$$m_{1n} + \dots + m_{r,n} = nM \quad m_{1,n-1} + m_{2,n-1} + \dots = (n-1)M \quad \dots \quad m_{11} = M \tag{2.12}$$

and using the GL$(n, \mathbb{C})$-invariant differential operators given by equation (3.5) of Klink and Ton-That (1989b) to diagonalise the $(M, \dots, M)$-weight space of $V'^{(m)}$, we can actually obtain the Gelfand–Žetlin basis vectors in this $(M, \dots, M)$-weight space.

The reciprocity theorem in Klink and Ton-That (1989b, corollary 2.9) also shows that the number of times the representation with signature (m) of GL$(n, \mathbb{C})$ occurs in $\mathscr{P}^{(M,\dots,M)}$ is given by the number of Gelfand–Žetlin tableaux satisfying the condition (2.12). This fact can be stated more precisely as follows.

Let $\mathscr{P}^{(M,\dots,M)}$ be the n-fold tensor product $V_N^{(M,0,\dots,0)} \otimes \dots \otimes V_N^{(M,0,\dots,0)}$. Then the dimension of the reducible representation of S_n modulo the irreducible representation of GL$(N, \mathbb{C})$ with signature (m) in the joint $\mathrm{S}_n \times$ GL$(N, \mathbb{C})$-module $\mathscr{P}^{(M,\dots,M)}$ is given by the number of Gelfand–Žetlin labels satisfying the weight condition (2.11) given above.

There are of course many well known formulae giving this dimension starting with the Littlewood–Richardson product rule; however, our formulation has a definite advantage in that it actually gives an explicit resolution of the S_n-multiplicities using these Gelfand tableaux and naturally leads to the calculation of Clebsch–Gordan and Racah coefficients.

To further decompose the reducible representation of S_n we remark that the subspace $\mathscr{T}^{(m)} \cap \mathscr{P}^{(M,\dots,M)}$ is usually called the zero-weight space of the GL$(n, \mathbb{C})$-module $\mathscr{T}^{(m)}$ and that several authors have investigated the decomposition of the zero-weight S_n-module of the GL$(n, \mathbb{C})$-module $V^{(m)}$. Such zero-weight spaces are discussed in the next section.

3. Representations of S_n on zero-weight spaces of GL$(N, \mathbb{C})$-modules

In the context of our problem, representations of the Weyl group of GL$(n, \mathbb{C})$ on zero-weight spaces of GL$(n, \mathbb{C})$-modules can be defined as follows.

We begin by identifying S_n with the subgroup of all permutation matrices in GL$(n, \mathbb{C})$ via the faithful representation $\sigma \to [\sigma] \in$ GL$(n, \mathbb{C})$; $\sigma \in \mathrm{S}_n$, where $[\sigma]$ is defined by $[\sigma]_{ij} = \delta_{i,\sigma(j)}$; $1 \leq i, j \leq n$, δ_{ij} being the Kronecker delta. An easy computation shows that the normaliser N_n of D_n in GL$(n, \mathbb{C})$ consists of all monomial matrices in GL$(n, \mathbb{C})$; i.e. matrices that have only one non-zero entry in each row and similarly in each column. It follows immediately that N_n is the semidirect product $\mathrm{S}_n \circledS \mathrm{D}_n$ (previously called K′, in section 1 so that S_n is identified with the Weyl group $\mathrm{W}_n = \mathrm{N}_n/\mathrm{D}_n$ of GL$(n, \mathbb{C})$). From section 2 we see that any polynomial representation of GL$(n, \mathbb{C})$ (respectively, GL$(N, \mathbb{C})$) can be realised as a GL$(n, \mathbb{C})$-module (respectively, GL$(N, \mathbb{C})$-module) in $\mathscr{F}(\mathbb{C}^{n \times N})$ of the form $V'^{(m)}$ (respectively $V^{(m)}$) of (2.4). A vector $F \in V'^{(m)}$

(respectively $V^{(m)}$) is called a weight vector with weight $(M_1, \ldots, M_n)$ (respectively, $(M_1, \ldots, M_N)$), M_i non-negative integers, if

$$L(d^{-1})F = d_1^{M_1} \ldots d_n^{M_n} \qquad \forall d \in \mathrm{D}_n$$

$$(\text{respectively, } R(d)F = d_1^{M_1} \ldots d_N^{M_N} \qquad \forall d \in \mathrm{D}_N).$$

Obviously the weight vectors in $V'^{(m)}$ (respectively in $V^{(m)}$) corresponding to the same weight (M) form a subspace of $V'^{(m)}$ (respectively $V^{(m)}$) which is called the weight space $V'^{(m)}_{(M)}$ (respectively $V^{(m)}_{(M)}$). If the weight (M) is such that $M_1 = \ldots = M_n = M$, then it is called the zero weight and the corresponding weight space is denoted by $V_0'^{(m)}$ (respectively, $V_0^{(m)}$). (This notation is justified by the fact that if we consider $V'^{(m)}$ (respectively $V^{(m)}$) as an irreducible $\mathrm{SL}(n, \mathbb{C})$-module (respectively $\mathrm{SL}(N, \mathbb{C})$-module) then for $d \in \mathrm{D}_n \cap \mathrm{SL}(n, \mathbb{C})$ (respectively, $\mathrm{D}_N \cap \mathrm{SL}(N, \mathbb{C})$) $d_1^M \ldots d_n^M = |d|^M = 1$ and $V_0'^{(m)}$ then consists of all vectors $F \in V'^{(m)}$ (respectively $V^{(m)}$) such that $L(d)F = F$ (respectively $R(d)F = F$).) From section 2 we can see that the zero-weight space of $V'^{(m)}$ is non-trivial if and only if $nM = |(m)| = m_1 + \ldots + m_r$, so that the zero-weight space of $V'^{(m)}$ is unique. Now from the abstract definition it is obvious that $V'^{(m)}$ is invariant under the Weyl group $\mathrm{W}_n = \mathrm{S}_n$; in fact, it is the only weight space of $V'^{(m)}$ that is W_n-invariant. Concretely we can see this as follows. Let $F \in V_0'^{(m)}$, $\sigma \in \mathrm{S}_n$ and $d \in \mathrm{D}_n$ then obviously $\sigma^{-1} d \sigma \in \mathrm{D}_n$ and therefore

$$\begin{aligned} L(d^{-1})L(\sigma)F &= L(\sigma(\sigma^{-1} d\sigma))F \\ &= L(\sigma)L(\sigma^{-1} d\sigma)F \\ &= L(\sigma)|d|^M F \\ &= |d|^M (L(\sigma)F). \end{aligned}$$

Thus $V_0'^{(m)}$ (respectively $V_0^{(m)}$) can be considered as an S_n-module (respectively, S_N-module). Now, if V is a $\mathrm{GL}(n, \mathbb{C})$-module we can decompose V in irreducible submodules and the zero-weight space V_0 of V is obviously the direct sum of the zero-weight spaces of its irreducible constituents. A particular case of special interest to us is the zero-weight space $\mathcal{T}_0^{(m)}$ of the isotypic component of $V'^{(m)}$ (or of $V^{(m)}$) which consists of p copies of the zero-weight space $V_0'^{(m)}$ if p is the multiplicity of $V'^{(m)}$ in $\mathcal{T}^{(m)}$.

Several authors have studied the S_n-module $V_0'^{(m)}$, especially the character $\chi_0^{(m)}$ of S_n. Kostant (1975) pointed out a very useful fact, namely that the value of $\chi_0^{(m)}$ on a Coxeter element μ of W_n is either 0 or ± 1 (in our context a Coxeter element can be taken as the permutation $(12, \ldots, n)$). Gutkin (1973) showed that $V'^{(m)}$ can be treated as an induced S_n-module. And finally Ariki *et al* (1985) gave a more direct method of decomposing $V_0^{(m')}$.

Using the theorems of Ariki *et al*, one can obtain in principle a spectral decomposition of the zero-weight S_n-module $V_0'^{(m)}$ by successively following the three steps given there. However, in practice, when n becomes large the computation becomes prohibitive, even with the help of computers. In section 5 we give a procedure, especially adaptable to computers, to compute the character $\chi_0^{(m)}$.

4. The ring of S_n-invariant differential operators on $\mathcal{F}(\mathbb{C}^{n \times N})$

As we have seen from sections 2 and 3, the decomposition of the $\mathrm{S}_n \times \mathrm{GL}(N, \mathbb{C})$-module $\mathcal{P}^{(M, \ldots, M)}$ reduces to the decomposition of submodules $\mathcal{T}_0^{(m)} = \mathcal{T}^{(m)} \cap \mathcal{P}^{(M, \ldots, M)}$ when (m) ranges over the signatures of all irreducible representations of $\mathrm{GL}(N, \mathbb{C})$ that occur in the $GL(N, \mathbb{C})$-module $\mathscr{P}^{(M,\ldots,M)}$. And this problem is in turn equivalent to the decomposition of the S_n-modules $V_0'^{(m)}$. But in general this decomposition is not multiplicity free. For example, the space $V_0'^{(8,4,0)}(\mathbb{C}^{3\times3})$ in the next section shows that the trivial representation of S_3 occurs once and the two-dimensional representation $\boxplus$ of S_3 occurs twice in this five-dimensional zero-weight space. We want, then, to find a canonical procedure to label the equivalent representations of S_n that occur in this $V_0'^{(m)}$; i.e. to resolve the multiplicity problem in $V_0'^{(m)}$. To do this we will construct a family of commuting S_n-invariant differential operators on $\mathscr{F}$ which when restricted to $V_0'^{(m)}$ will diagonalise $V_0'^{(m)}$ and whose distinct eigenvalues can serve as labels of the irreducible representations of S_n that occur in $V_0'^{(m)}$. We proceed as follows.

Let $\mathscr{U}$ denote the universal enveloping algebra of differential operators generated by the Lie algebra basis $\{L_{\alpha\beta},\ 1 \leqslant \alpha, \beta \leqslant n\}$ of $GL(n, \mathbb{C})$ given by (2.10); then $\mathscr{U}$ also acts on $\mathscr{F}$. A differential operator $U \in \mathscr{U}$ is said to be *S_n-invariant* if $UL(\sigma) = L(\sigma)U$ (or equivalently $L(\sigma)\ UL(\sigma^{-1}) = U$) for all $\sigma \in S_n$. We have the following theorem.

Theorem. The S_n-invariant differential operators form a subalgebra of $\mathscr{U}$. This subalgebra is finitely generated and a set of generators can be given by elements of the form $U \equiv U(L_{\alpha\beta}) \in \mathscr{U}$ such that $U(L_{\sigma(\alpha),\sigma(\beta)}) = U(L_{\alpha,\beta})$ for all $\sigma \in S_n$ and all $\alpha, \beta = 1, \ldots, n$.

Proof. Let $gl(n, \mathbb{C})$ denote the Lie algebra spanned by the basis element $L_{\alpha\beta}$, $1 \leqslant \alpha$, $\beta \leqslant n$. Then each element X in $gl(n, \mathbb{C})$ can be written uniquely as

$$X = \sum_{\alpha,\beta=1}^{n} X_{\alpha\beta} L_{\alpha\beta} \qquad X_{\alpha\beta} \in \mathbb{C}$$

so that X can be identified with the matrix $[X_{\alpha\beta}] \in \mathbb{C}^{n\times n}$. Let S denote the symmetric algebra of all polynomial functions of $\mathbb{C}^{n\times n}$. Let T denote the coadjoint representation of $GL(n, \mathbb{C})$ in S defined by

$$[T(h)p](X) = p(h^{-1}Xh) \qquad h \in GL(n, \mathbb{C}), p \in S.$$

Now, it was shown in Klink and Ton-That (1988) that under the canonical isomorphism Φ of S onto $\mathscr{U}$ (cf Dixmier 1974, chapter 3) that an element $\Phi(p)$ in $\mathscr{U}$ is S_n-invariant if and only if $p([\sigma]^{-1}X[\sigma]) = p(X)$ for all $\sigma \in S_n$. If we identify a permutation σ with a matrix in $GL(n, \mathbb{C})$ via the equation

$$[\sigma]_{ij} = \delta_{i,\sigma(j)} \qquad 1 \leqslant i, j \leqslant n$$

then an easy computation shows that

$$([\sigma^{-1}]X[\sigma])_{ij} = X_{\sigma(i),\sigma(j)}.$$

It follows that a polynomial $p(X)$ is S_n-invariant, i.e. $p(\sigma^{-1}X\sigma) = p(X)\ \forall \sigma \in S_n$, if and only if expanded in the variables X_{ij}, $1 \leqslant i, j \leqslant n$, it is invariant under any permutation of the indices (i, j).

Let E_{ij} denote the matrix with 1 in the (i, j) entry and 0 elsewhere and let $\sigma \in S_n$; then

$$[\sigma^{-1}E_{ij}\sigma]_{kl} = \delta_{i\sigma(k)}\delta_{j\sigma(l)}.$$

If $\rho : S_n \to GL(\mathbb{C}^{n\times n}) \approx GL(n^2, \mathbb{C})$ is the representation of σ in $\mathbb{C}^{n\times n}$ defined by

$$\rho(\sigma)(E_{ij}) = \sigma^{-1}E_{ij}\sigma \qquad \forall E_{ij}$$

then we have an embedding of S_n into S_{n^2}. This action ρ extends in an obvious manner to a linear action on $S(\mathbb{C}^{n\times n})$. It follows from Springer (1977) that S_n is a finite reflection group and hence by a theorem of Chevalley (1955) the ring of S_n-invariants is finitely generated by algebraically independent homogeneous elements. □

Remark. The centre of $\mathcal{U}$ consisting of differential operators which are $GL(n, \mathbb{C})$-invariants is obviously contained in the subalgebra of all S_n-invariant differential operators. Therefore, if we let $[L]$ denote the matrix $[L_{\alpha\beta}]$, $1 \leq \alpha, \beta \leq n$ then the non-commutative trace operators $\mathrm{Tr}([L]^k)$ are $GL(n, \mathbb{C})$-invariant, and hence S_n-invariant. It is easy to show that there are many S_n-invariant differential operators that are not $GL(n, \mathbb{C})$-invariant; e.g. for $n = 3$

$$U^{(4)} = \sum_{\sigma \in S_3} L_{\sigma(1)\sigma(2)} L_{\sigma(2)\sigma(3)} L_{\sigma(3)\sigma(2)} L_{\sigma(2)\sigma(1)}. \tag{4.1}$$

5. Computing S_n characters on the $GL(n, \mathbb{C})$-modules $V'^{(m)}$

As mentioned previously, there are several closed-form expressions giving the character of S_n on $V_0'^{(m)}$. However, these formulae are sufficiently complicated that they are not very useful from a computational point of view. In this section we will present a method for computing the S_n character which is easy to implement from a computational point of view, but is too complicated to be able to write down in closed form. We conclude this section with two examples, the first of which uses an S_n-invariant differential operator of the form discussed in the previous section to break the multiplicity.

In Klink and Ton-That (1989a, b) we showed how to obtain a Gelfand–Žetlin basis for $V_0'^{(m)}$ from its highest-weight vector. It follows that if a Gelfand state $h_{[k]}^{(m)}$ is in $V_0'^{(m)}$ and if $\sigma \in S_n$ then

$$L(\sigma) h_{[k]}^{(m)} = \sum_{[k']} D_{[k'][k]}^{(m)}(\sigma) h_{[k']}^{(m)} \tag{5.1}$$

where $D_{[k'][k]}^{(m)}$ are matrix elements of a reducible representation of S_n and the sum is over all Gelfand patterns $[k']$ of weight $(M, \ldots, M)$. Since the $h_{[k]}^{(m)}$ form an orthogonal basis, it is possible to write down the S_n matrix elements as

$$D_{[k'][k]}^{(m)}(\sigma) = \frac{\langle h_{[k']}^{(m)}, L(\sigma) h_{[k]}^{(m)} \rangle}{\| h_{[k']}^{(m)} \|^2}.$$

The character $\chi_0^{(m)}$ is then given by

$$\begin{aligned} \chi_0^{(m)}(\sigma) &= \mathrm{Tr}(D^{(m)}(\sigma)) \\ &= \sum_{[k]} D_{[k][k]}^{(m)}(\sigma) \\ &= \sum_{[k]} \frac{\langle h_{[k]}^{(m)}, L(\sigma) h_{[k]}^{(m)} \rangle}{\| h_{[k]}^{(m)} \|^2}. \end{aligned} \tag{5.2}$$

Since the $h_{[k]}^{(m)}$ are polynomials in Z and the norms of $h_{[k]}^{(m)}$ have been computed (Ho Pei-Yu 1966), it is straightforward though tedious to compute the inner products in (5.2) and hence to get the character $\chi_0^{(m)}$.

In the remaining part of this section we use (5.2) to compute a number of characters to illustrate the theory.

First we show that the Schur–Weyl duality theorem is a special case of our procedure. To see this consider the n-fold tensor product of the representation of $GL(N, \mathbb{C})$ of signature $(1, 0, \ldots, 0)$. To get the multiplicity of the representation of $GL(N, \mathbb{C})$ with signature $(m_1, \ldots, m_N, 0, \ldots, 0)$ if $N \leq n$ or $(m_1, \ldots, m_n, \ldots, 0)$ if $n < N$ we consider the weight condition $m_1 + \ldots + m_n = nM = n$ (since $M = 1$) and the chain of subgroups $GL(N, \mathbb{C}) \supset GL(N-1, \mathbb{C}) \supset \ldots \supset GL(1, \mathbb{C})$. In $V^{\prime(m_1, \ldots, m_r)}$ $(r = \min(n, N))$ the Gelfand–Žetlin basis elements are $h_{[k]}^{(m)}$, where, for example, in the case $n < N$

$$[k] = \begin{bmatrix} m_{1,N} & m_{2,N} & \cdots & m_{n,N} & \cdots & 0 & \cdots & 0 \\ & m_{1,N-1} & m_{n,N-1} & \cdots & m_{n,N-1} & \cdots & 0 & \\ & \ddots & \ddots & & & & & \\ & & m_{12} & m_{22} & & & & \\ & & & m_{11} & & & & \end{bmatrix}$$

are Gelfand tableaux which satisfy the weight condition

$$\begin{aligned} & m_{iN} = m_i \qquad i \leq i \leq n \\ & m_{1,N-1} + \ldots + m_{n,N-1} = n-1 \\ & m_{1,N-2} + m_{2,N-2} + \ldots = n-2 \\ & \vdots \\ & m_{12} + m_{22} = 2 \\ & m_{11} = 1. \end{aligned} \tag{5.3}$$

Moreover, each tuple $(m_{1i, 2i, \ldots})$ is the signature of an irreducible representation of $GL(i, \mathbb{C})$, $1 \leq i \leq N$. S_n acts on the space $V_0^{\prime(m)}$, and according to the statement in section 2 the dimension of this representation is given by the number of tableaux $[k]$ satisfying the condition (5.3). The $N-1$ tuple $(m_{1,N-1}, \ldots, m_{n,N-1}, 0, \ldots, 0)$ with $m_{1,N-1} + \ldots + m_{n,N-1} = n-1$ corresponding to the signature of an irreducible representation of $GL(n-1, \mathbb{C})$ in the $n-1$ tensor product

$$\underbrace{(1, 0, \ldots, 0) \otimes \ldots \otimes (1, 0, \ldots, 0)}_{n-1}$$

in turn induces a representation of S_{n-1}. Thus, by iterating this process we have a chain of subgroups $S_n \supset S_{n-1} \supset \ldots \supset S_1$ acting on $V_0^{\prime(m)}$. We prove by induction on i that each tuple $(m_{1i}, m_{2i}, \ldots)$ corresponds to a Young diagram of S_i. From (5.3) this is obviously true for $i = 1$. Assume this is true for all integers $\leq i$ and consider the subtableau of $[k]$ of the form

$$\begin{bmatrix} m_{1,i+1} & m_{2,i+1} & \\ m_{1,i} & m_{2i} & \\ \ddots & \ddots & \\ & m_{1,2} & m_{22} \\ & m_{11} & \end{bmatrix}$$

where by the inductive hypothesis each $(m_{1i}, m_{2i}, \ldots)$ corresponds to Young diagram of an irreducible representation of S_i. The betweeness relations $m_{1,i+1} \geq m_{1i} \geq m_{2,i+1} \geq m_{2,i} \ldots$ and the conditions $m_{1,i+1} + m_{2,i+1} + \ldots = i+1$, $m_{1,i} + m_{2,i} + \ldots = i$ show that by the branching law for irreducible representations of the symmetric groups (cf Hammermesh 1962; especially equation (7.52), p 210) that the tuple $(m_{1,i+1}, m_{2,i+1}, \ldots)$ is the

Young tableau of an irreducible representations of S_{r+1}. It follows that the representation of S_n on $(V'^{(m)})_0$ is irreducible and its signature is the Young diagram $(m_1, m_2, \ldots, m_n, \ldots)$. This is exactly the Schur–Weyl duality theorem.

Example 1. Consider the threefold tensor product $V^{(4,0,0)} \otimes V^{(4,0,0)} \otimes V^{(4,0,0)} \approx \mathscr{P}^{(4,4,4)}$ of GL(3, $\mathbb{C}$); the multiplicity of $V^{(8,4,0)}$ in this tensor product is 5 and the Gelfand–Žetlin basis for $(V'^{(8,4,0)})_0$ is the set

$$\left\{ h\begin{pmatrix} 8\ 4\ 0 \\ 8\ 0 \\ 4 \end{pmatrix}, h\begin{pmatrix} 8\ 4\ 0 \\ 7\ 1 \\ 4 \end{pmatrix}, h\begin{pmatrix} 8\ 4\ 0 \\ 6\ 2 \\ 4 \end{pmatrix}, h\begin{pmatrix} 8\ 4\ 0 \\ 5\ 3 \\ 4 \end{pmatrix}, h\begin{pmatrix} 8\ 4\ 0 \\ 4\ 4 \\ 4 \end{pmatrix} \right\}.$$

By proposition 5.4 of Ariki *et al* (1985), the symmetric representation of S_3 must occur in the reduction of $(V'^{(8,4,0)})_0$. By a theorem of Kostant (1975) $\chi_0^{(8,4,0)}(\mu)$ is either 0, 1, or -1. From these facts it is easy to deduce that the symmetric representation ρ_s must occur in $(V'^{(8,4,0)})_0$ once, and the two-dimensional representation ρ_2 must occur in $(V'^{(8,4,0)})_0$ twice. To resolve this multiplicity problem, we apply the S_3-invariant differential operator $U^{(4)}$ of (4.1) to $V'^{(8,4,0)}$. The eigenvalues of this operator are 168, $270+6\sqrt{193}$ and $270-6\sqrt{193}$. The eigenvalue 168 labels the symmetric representation ρ_s, the eigenvalue $270+6\sqrt{193}$, which occurs with multiplicity 2, labels one copy of the two-dimensional representation ρ_2 while $270-6\sqrt{193}$ labels the other two-dimensional representation of ρ_2. (The symbolic manipulation program of Klink and Ton-That (1988) was used to compute the eigenvalues of the operator $U^{(4)}$.)

Table 1. Results of example 2.

$\Lambda \to S_3$ representations	Classes			Multiplicity		
	e	(12)(3)	(123)	$n(S)$	$n(A)$	$n(2)$
χ_s	1	1	1			
χ_A	1	-1	1			
χ_2	2	0	-1			
$\chi_0^{2M,M,0}$, $M=1$	2	0	-1	0	0	1
3	4	0	1	1	1	1
5	6	0	0	1	1	2
7	8	0	-1	1	1	3
9	10	0	1	2	2	3
11	12	0	0	2	2	4
13	14	0	-1	2	2	5
⋮	⋮	⋮	⋮	⋮	⋮	⋮
2	3	1	0	1	0	1
4	5	1	-1	1	0	1
6	7	1	1	2	1	2
8	9	1	0	2	1	3
10	11	1	-1	2	1	4
12	13	1	1	3	2	4
14	15	1	0	3	2	5
16	17	1	-1	3	2	6

Example 2. In this example we consider the threefold tensor product of an arbitrary irreducible representation of SU(2) labelled by an integer or half-integer j. This representation corresponds to the representation of $GL(2, \mathbb{C})$ of signature $(M, 0)$ with $M = 2j$. Then the number of times the representation j of SU(2) occurs in the tensor product is given by the number of Gelfand patterns

$$\begin{pmatrix} 2M & & M & & 0 \\ & M+l & & M-l & \\ & & M & & \end{pmatrix} \qquad 0 \leq l \leq M$$

which is obviously equal to $M+1$, so that the Gelfand-Žetlin basis for $(V^{\prime j})_0 = (V^{\prime(2M,M)})_0$ is the set

$$\left\{ h_{\begin{pmatrix} 2M & & M & & 0 \\ & M+l & & M-l & \\ & & M & & \end{pmatrix}} \right\}_{l=0,\ldots,M}.$$

A concrete realisation of these basis elements in the Fock space $\mathscr{F}(\mathbb{C}^{3\times 2})$ can be given in the following way. Let $Z \in \mathbb{C}^{3\times 2}$ and denote by $\Delta_1^i(Z)$ the $(i, 1)$-entry of Z, $1 \leq i \leq 3$. Let $\Delta_{12}^{ij}(Z)$ denote the minor of Z formed by the rows i, j and the columns 1, 2. Then an orthogonal basis for $(V^{1j})_0$ is

$$h_{\begin{pmatrix} 2M & & M & & 0 \\ & M+l & & M-l & \\ & & M & & \end{pmatrix}}(Z) = [\Delta_1^3(Z)\Delta_{12}^{12}(Z)]^{M-l}[\Delta_1^2(Z)\Delta_{12}^{13}(Z) + \Delta_1^1(Z)\Delta_{12}^{23}(Z)]^l.$$

Using equation (5.2) then gives $\chi(e) = M+1$ and $\chi((12)(3)) = \Sigma_{l=0}^{M}(-1)^{M-l}$; the computation of $\chi(123)$ is more complicated and will not be given here. From the expression for multiplicity we get the results presented in table 1.

6. Conclusion

We have given a computationally effective procedure to resolve the multiplicity of irreducible representations of the symmetric group S_n occurring in repeated tensor products of $U(N)$ $(GL(N, \mathbb{C}))$ representations of the form $(M0\ldots0)$. We are developing symbolic manipulation programs which generate the Gelfand-Žetlin tableaux and the resulting reducible characters of S_n; in fact the examples at the end of section 5 were carried out using such programs.

If in a tensor product decomposition the multiplicity of an S_n irreducible representation is greater than 1, invariant operators from the enveloping algebra of $GL(n, \mathbb{C})$ which commute with S_n may be introduced to break the S_n multiplicity. In section 4 we showed that such invariant operators are finitely generated, and gave their general form. Given an invariant operator, it is still necessary to diagonalise the operator; we are also developing computer programs to compute the relevant eigenvalues and eigenvectors of the multiplicity-breaking invariant operators.

All of the operations for obtaining basis elements that transform irreducibly under S_n and $GL(N, \mathbb{C})$ start with polynomials associated with Gelfand-Žetlin tableaux. That is, for each Gelfand-Žetlin tableau there is a polynomial element in the zero-weight space, the intersection of the isotypic component of $GL(N, \mathbb{C}) \times \ldots \times GL(N, \mathbb{C})$ and $GL(n, \mathbb{C}) \times GL(N, \mathbb{C})$. These polynomials form a basis for a reducible representation of S_n. By taking suitable linear combinations of these basis elements so that they transform irreducibly under S_n, it is possible to compute Clebsch-Gordan coefficients which carry S_n labels rather than intermediate coupling labels, as is usually the case when computing Clebsch-Gordan coefficients of n-fold tensor products in a stepwise fashion. Such symmetric group labelled Clebsch-Gordan coefficients are obtained by taking the inner product of the symmetric group labelled basis polynomials with tensor product basis polynomials; in our procedure computing an inner product means differentiating the polynomials in a certain way (see Klink and Ton-That 1989a). Clebsch-Gordan coefficients carrying S_n irreducible representation labels will be discussed in a subsequent publication.

In this paper we have only examined the symmetric group content of n-fold tensor products of $U(N)$ representations of the form $(M, 0, \ldots, 0)$. It is possible to generalise our results to arbitrary irreducible representations of the form $(M_1, M_2, \ldots, M_r, 0, \ldots, 0) \equiv (M)$ with $1 \leqslant r \leqslant N$. In this case the dual to $GL(N, \mathbb{C}) \times \ldots \times GL(N, \mathbb{C})$ is $GL(r, \mathbb{C}) \times \ldots \times GL(r, \mathbb{C})$, and when the outer product group is restricted to the diagonal subgroup, the dual becomes $GL(rn, \mathbb{C})$. The Fock space is thus $\mathscr{F}(\mathbb{C}_{rn \times N})$ and can be decomposed into isotypic components $\mathscr{T}_{(M)}$ or $\mathscr{T}(m)$, where $(m) = (m_1, \ldots, m_N)$ is an irreducible representation of $GL(N, \mathbb{C})$ and $GL(rn, \mathbb{C})$. Using operators from the Lie algebra of $GL(rn, \mathbb{C})$, elements from the irrducible representation space $V^{(m)}$ of $GL(N, \mathbb{C})$ can be mapped into $\mathscr{T}_{(M)}$. Then the symmetric group S_{rn} leaves the zero-weight space $\mathscr{T}_{(m)} \cap \mathscr{T}_{(M)}$ invariant. Since S_n is a subgroup of S_{rn}, it is clear that the reducible representation of S_{rn} on $\mathscr{T}_{(m)} \cap \mathscr{T}_{(M)}$ is also a reducible representation of S_n. Procedures for obtaining the multiplicities and basis elements of irreducible representations of S_n will be given in a future paper.

In a similar fashion it is also possible to obtain the symmetric group content of n-fold tensor products of repeated representations of other compact groups, considered as subgroups of $GL(N, \mathbb{C})$. For example, the dual group to $SO(N)$ is the symplectic group. If (M) is an irreducible representation of $GL(N, \mathbb{C})$ the irreducible representations of $SO(N)$ contained in (M) are obtained from lowering operators in the Lie algebra of the symplectic group which annihilate elements in the space labelled by (M). n-fold tensor products of $SO(N)$ come from the diagonal subgroup of $SO(N) \times \ldots \times SO(N)$. The dual to the diagonal subgroup $SO(N)$ will be a big symplectic group containing the outer product symplectic groups. Using raising operating from the Lie algebra of the symplectic group, elements can be mapped into the n-fold tensor product space, which again carries a reducible representation of S_n. In fact a generalisation of the Schur-Weyl duality theorem for the orthogonal and symplectic groups was given in Wenzl (1988) but one has to consider Brauer's centraliser algebras instead of groups (S_n in the classical case).

Thus it is clear that as soon as the dual to a compact subgroup of $GL(N, \mathbb{C})$ is known, it is possible to find the S_n content of n-fold tensor products of repeated representations of that subgroup. Methods for carrying out such procedures for various compact groups will be given in subsequent publications.

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